Ballistics of a No. 3 Spherical Shell with Illuminant

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Abstract: *Shot experiments were carried out using a No. 3 spherical shell with an illuminant. The three dimensional trajectory of the flying shell was obtained by tracing the shell with two high-speed video cameras in different orientations and by analyzing the recorded video picture. The 3DSTAR1 computer code was developed for calculating the three dimensional trajectory of a flying shell with high accuracy. The optimal drag coefficient, wind speed and wind direction were estimated using the 3DSTAR1 code by fitting the calculated trajectory to the experimental one for a No. 3 shell.*

Keywords: *ballistics, no. 3 shell, trajectory*

Introduction

A firework shell is shot from a mortar, rises into the air, bursts in the sky and releases burning stars. The stars fly into the sky and form a composition like a blooming flower or bunch of flowers. The shell is set in the mortar on top of a lifting charge. The lifting charge is ignited and the shell is shot into the air by the pressure developed by the combustion gas of the lifting charge.

The ballistics of the shell expelled from the muzzle of the mortar are affected by various factors such as muzzle velocity, air drag, shot angle, mass of lifting charge, wind direction and speed, and so on. In terms of the design and safety of the shell shot, it is important to know the basic ballistics of the shell.

Shimizu[1,2] has carried out shot experiments using spherical shells and analyzed the results. Kosanke and Kosanke[3] have performed theoretical modeling of the ballistics of shells using Shimizu's experimental data. Recently, Iida *et al*.[4] have carried out a shot experiment using several sizes of spherical shells. Eckhardt and Andre,[5] and Speer[6] have calculated the trajectories of spherical firework shells in order to investigate an accident at a public fireworks display in 1997. Mercer[7] has modeled the aerodynamics of propelled aerial shells. Schneider and Schneider[8] performed ballistic trajectory calculations to investigate the relationship between the launch elevation of dud aerial firework devices and ground impact distances.

The objectives of this paper are as follows:

(1) A No. 3 spherical shell is shot from a mortar and the trajectory of the shells is observed from different directions by two high-speed video cameras, and the results are three-dimensionally analyzed.

(2) A three-dimensional theoretical model of the ballistics of the shell is developed, a theoretical calculation is carried out, and the effect of wind direction and speed, air drag, shot angle and so on are examined.

Experimental

Materials

No. 3 spherical shells with an illuminant for tracing the trajectory were made by the Sunaga Fireworks Company. The lifting charge and electric match were made by the Nippon Kayaku Company. The mass and diameter of the shells and the mass of lifting charges in this experiment are listed in Table 1 along with the observed muzzle velocity.

Apparatus

The mortar for the No. 3 shells was made of steel and the dimensions of the mortar are shown in Figure 1.

The trajectory of an expelled shell was traced

Table 1 *Mass and diameter of the shell, mass of lifting charge, and muzzle velocity.*

Run	Mass (kg)	Diameter (m)	Lifting charge (g)	Muzzle velocity (m s^{-1})
1	0.255	0.083	10	73
2	0.258	0.083	8	57
3	0.250	0.083	6	30
4	0.250	0.083	6	23
5	0.241	0.084	6	33
6	0.241	0.083	6	34

by two high-speed video cameras (Phantom VR-V4.2) with a frame rate of 150 frames per second. The locations of the two cameras are shown in Figure 2.

Procedure

The mortar was set on the ground vertically. The lifting charge and electric match wrapped in thin paper were put in the bottom of the mortar. Then a No. 3 shell was placed on the lifting charge. The electric match was ignited by turning on an electric current. The lifting charge burned, pressure developed, the shell moved upwards and was expelled into the air. The trajectory of the shell in the air was recorded by the high-speed cameras.

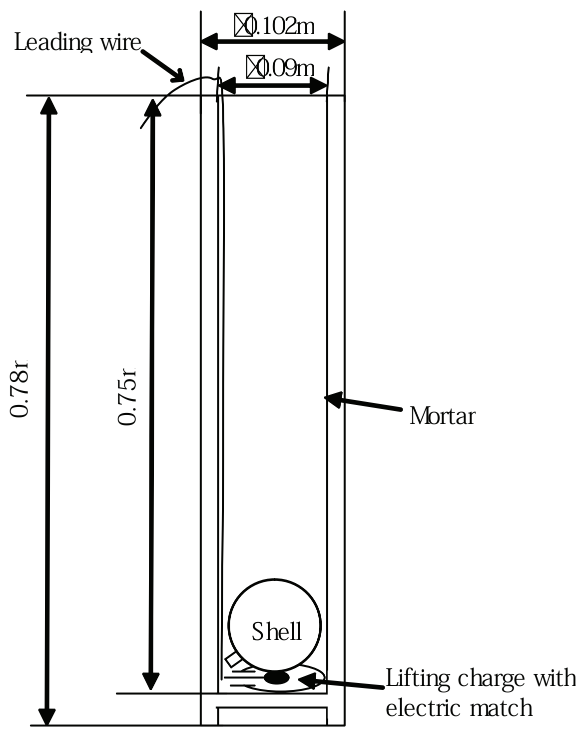

Figure 1 *Dimensions of the mortar for No. 3 shells.*

Three dimensional analysis of experimental data

The position of the shell flying in the air is analyzed using two cameras located as in Figure 2. Two three-dimensional rectangular coordinates are provided for the locations of the cameras. The muzzle of the mortar is the origin of the coordinates. The perpendicular direction is the *z* axis of the coordinates. The two horizontal coordinates are *X*–*Y* and *x*–*y*, and camera 1 is on the *X* axis of the *X*–*Y* coordinates and camera 2 is on the *y* axis of the *x*–*y* coordinates as shown in Figure 2. The *Y* coordinate of the shell in the *X*–*Y* coordinates is recorded by camera 1 and the *x* coordinate in the *x*–*y* coordinates is recorded by camera 2. The relationships of the two coordinates

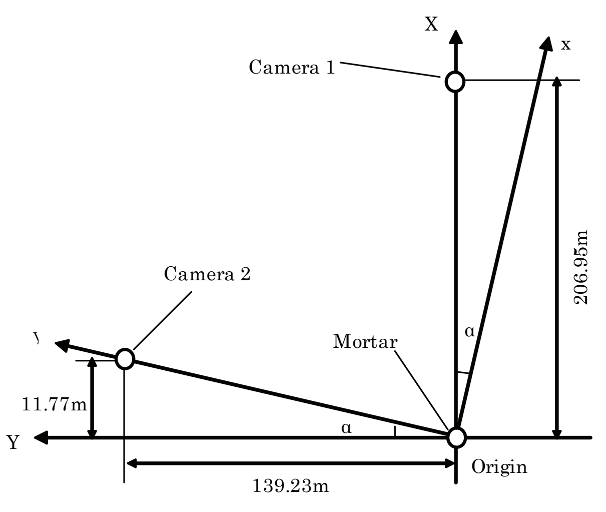

Figure 2 *Location of high-speed cameras.*

are as follows.

$$x = X\cos\alpha - Y\sin\alpha \qquad (1)$$

$$y = X\sin\alpha + Y\cos\alpha \qquad (2)$$

The coordinate y is calculated from the angle α, and the recorded x and Y values, using equations (1) and (2).

Finally, the spatial position (x, y, z) of the firework shell in one set of three-dimensional rectangular coordinates is obtained. These coordinates were converted to the real distances, and the real spatial position of the shell at a given time is calculated.

Three-dimensional model for ballistics of a shell

Equations of motion

As a shell flies into the air, a complex aerodynamic drag force acts on it. The force will depend on the density of the air, the viscosity of the air, the shape and surface roughness of the shell, etc. To simplify the problem, the following assumptions are made.

(1) The shell does not spin in flight.

(2) The shell is spherical.

(3) The air density and air viscosity are constant.

(4) The speed and the direction of the wind are constant.

The vectors of position, velocity and acceleration of the shell, and the force acting on the shell, are expressed by rectangular coordinates. For example, the vector of the position of a shell is expressed by rectangular coordinates as shown in Figure 3.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$x = r\sin\beta_1\cos\alpha_1 \qquad y = r\sin\beta_1\sin\alpha_1$$

$$z = r\cos\beta_1$$

$$r^2 = x^2 + y^2 + z^2 \qquad (3)$$

The motion equation of a spherical shell is

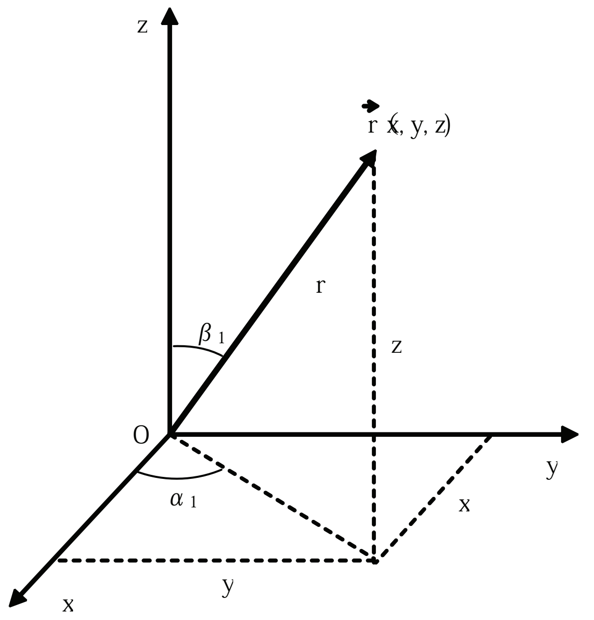

Figure 3 *Three dimensional coordinates of positional vector* $\vec{r}$.

$$\tan\alpha_1 = \frac{y}{x}$$

$$\tan\beta_1 = \frac{\sqrt{x^2 + y^2}}{z}$$

Here, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors of the coordinates x, y and z, respectively.

α_1 is the angle between the shadow of the vector $\vec{r}$ in the x–y plane and the coordinate x, and β1 is the angle between the vector $\vec{r}$ and the coordinate z

expressed as follows:

$$m\frac{d\vec{v}}{dt} = m\vec{g} + \vec{F}_D + \vec{F}_B \qquad (4)$$

Here, $\vec{g}$, $\vec{v}$, $\vec{F}_D$ and $\vec{F}_B$ are the vectors of acceleration under gravity, velocity of the shell, air drag and buoyancy, respectively. m is the mass of the shell.

Furthermore,

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad (5)$$

$$\vec{F}_D = -\frac{\rho A C_D}{2} \cdot |\vec{u}| \cdot \vec{u} = -k|\vec{u}| \cdot \vec{u} \quad (6)$$

$$\vec{F}_B = -\frac{\rho}{\rho_p} m\vec{g} \quad (7)$$

Here, ρ, ρ_p, A and C_D are air density, density of the shell, cross-sectional area of the shell and drag coefficient of air, respectively. $\vec{r}$ and $\vec{u}$ are positional vector of the shell and relative velocity vector between air and the shell.

The motion velocity of a shell:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \quad (8)$$

Here,

$$v_x = |v|\sin\beta_2\cos\alpha_2$$
$$v_y = |v|\sin\beta_2\sin\alpha_2$$
$$v_z = |v|\cos\beta_2$$
$$v^2 = {v_x}^2 + {v_y}^2 + {v_z}^2$$
$$\tan\alpha_2 = \frac{v_y}{v_x}$$
$$\tan\beta_2 = \frac{\sqrt{{v_x}^2 + {v_y}^2}}{v_z}$$

The wind velocity vector $\vec{w}$ is constant,

$$\vec{w} = w_x\vec{i} + w_y\vec{j} + w_z\vec{k} \quad (9)$$

Here,

$$w_x = |w|\sin\beta_3\cos\alpha_3$$
$$w_y = |w|\sin\beta_3\cos\alpha_3$$
$$w_z = |w|\sin\beta_3\cos\alpha_3$$

Relative velocity between the shell and air is:

$$\vec{u} = \vec{v} - \vec{w} = (v_x - w_x)\vec{i} + (v_y - w_y)\vec{j} + (v_z - w_z)\vec{k} \quad (10)$$

$$|\vec{u}| = |\vec{v} - \vec{w}| = \sqrt{(v_x - w_x)^2 + (v_y - w_y)^2 + (v_z - w_z)^2} \quad (11)$$

Therefore, the motion equations of a flying shell in the air are:

$$\frac{d\vec{v}}{dt} = (1 - \frac{\rho}{\rho_p})\vec{g} - k|\vec{v} - \vec{w}|(\vec{v} - \vec{w}) \quad (12)$$

In the three dimensional rectangular coordinate shown in Figure 3, the gravity acceleration can be expressed as follows:

$$\vec{g} = -g\vec{k}$$

And the equations of motion are:

$$\frac{dv_x}{dt} = -k|\vec{v} - \vec{w}| \cdot (v_x - w_x) \quad (13)$$

$$\frac{dx}{dt} = v_x \quad (14)$$

$$\frac{dv_y}{dt} = -k|\vec{v} - \vec{w}| \cdot (v_y - w_y) \quad (15)$$

$$\frac{dy}{dt} = v_y \quad (16)$$

$$\frac{dv_z}{dt} = -(1 - \frac{\rho}{\rho_p})g - k|\vec{v} - \vec{w}| \cdot (v_z - w_z) \quad (17)$$

$$\frac{dz}{dt} = v_z \quad (18)$$

Here, $k = \frac{\rho A C_D}{2m}$; when the shell is a sphere, $k = \frac{3\rho C_D}{4\rho_p D_p}$, where D_p is the diameter of the sphere.

Numerical calculation of motion equations

It is difficult to integrate equations (13)–(18) directly. Therefore, numerical calculation of equations (13)–(18) was carried out using the fourth-order Runge–Kutta method in this work.

Results and Discussion

Calculation accuracy of the 3DSTAR1 code

The numerical calculation results by 3DSTAR1 were compared with the analytical solutions, and the accuracy of the former was validated.

At first, considering the motion of the shell expelled perpendicularly into the sky without air drag, the perpendicular velocity V_z and position z were calculated using 3DSTAR1. The muzzle velocity of the expelled shell was assumed to be 119 m s^{-1} and the calculated results are shown in Figure 10.

On the other hand, V_z and z can be obtained analytically as follows:

$$V_z = V_0 - g \cdot t \tag{19}$$

$$z = z_0 + V_0 \cdot t - \frac{1}{2} g \cdot t^2 \tag{20}$$

As shown in Figure 4, the results from the 3DSTAR1 numerical calculation agreed completely with the analytical solution of equations (19) and (20).

Effect of time interval on calculation accuracy of 3DSTAR1 code

Kosanke and Kosanke[3] published "Computer Modeling of Aerial Shell Ballistics", and validated the results using Shimizu's experimental data. The effect of the time interval on the accuracy of the 3DSTAR1 code was examined using the same data:

Muzzle velocity of the shell: 118.95 m s^{-1}

Diameter of the shell: 0.17125 m

Mass of the shell: 2.1111 kg

Angle of the mortar: 0°

Wind speed: 0 m s^{-1}

Drag coefficient: 0.36

When the drag coefficient is constant, the analytical solutions of the equations of motion of the shell expelled perpendicularly are as follows

Time to apogee $$t_1 = \frac{1}{\sqrt{gk}} \cdot \tan^{-1}\left(\sqrt{\frac{k}{g}} \cdot V_0\right) \tag{21}$$

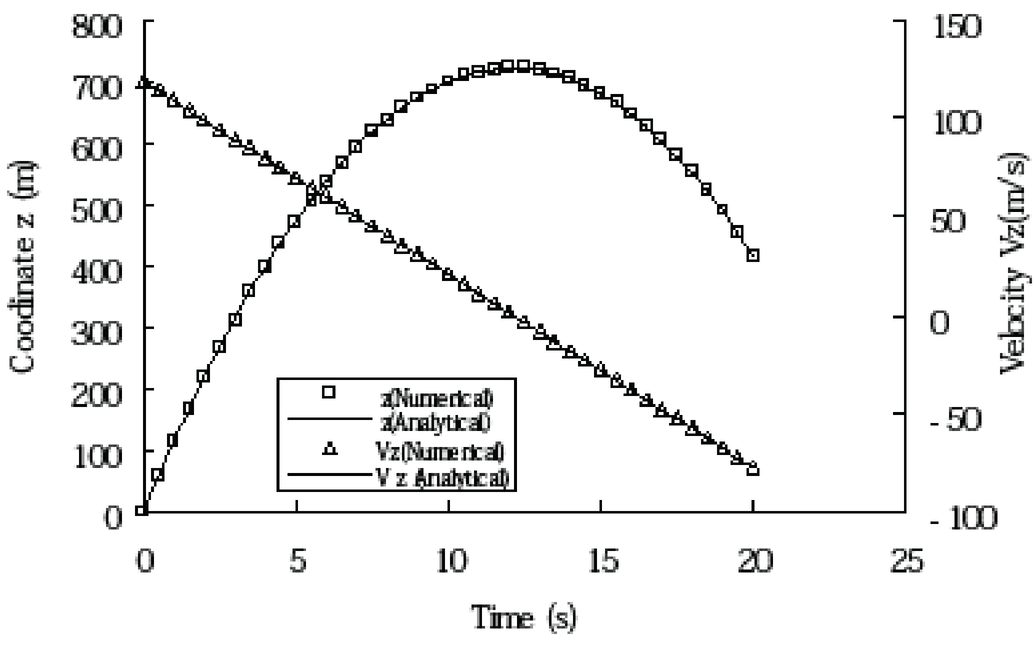

Figure 4 *Comparison of 3DSTAR1 calculation and analytical solution. Without air drag and with 119 m s^{-1} muzzle velocity.*

Apogee height $$Z_1 = \frac{1}{2k} \cdot \ln\left(1 + \frac{k}{g} V_0^2\right) \quad (22)$$

Velocity to impact $$V_2 = -\sqrt{\frac{g}{k}} \cdot \left(1 - e^{-2kZ_1}\right) \quad (23)$$

Time to impact $$t_2 = t_1 - \frac{1}{\sqrt{gk}} \cdot \tanh^{-1}\left(\sqrt{\frac{k}{g}} \cdot V_2\right) \quad (24)$$

Here, $k = \frac{3\rho C_D}{4\rho_p D_p}$ and g, ρ, C_D, ρ, D_p and V_0 are acceleration under gravity, air density, air drag coefficient, and density, diameter and muzzle velocity of the shell, respectively. The velocity of the shell is positive when the shell moves upward, and negative when it moves downward.

The results are shown in Table 2 and Figure 5.

It can be seen from Figure 5 that even if the time interval is 1 s, the computer code can give high enough calculation accuracy.

Results of observed and calculated three dimensional trajectory

The video pictures of the motion of the shell in the air were recorded by two cameras facing in

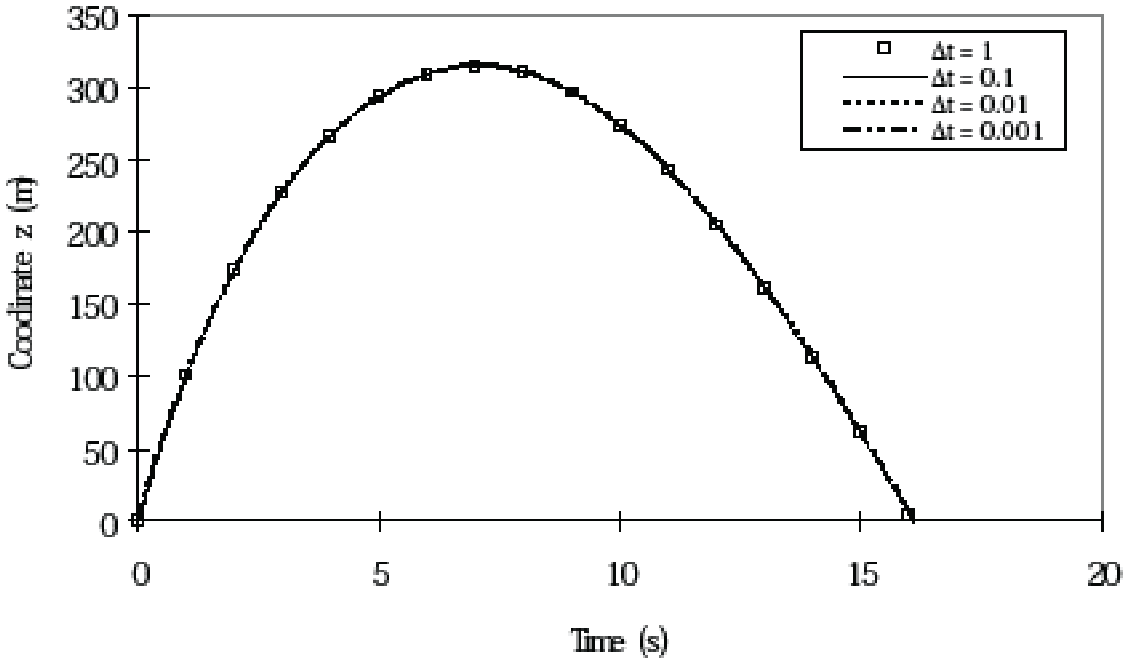

Figure 5 *Calculated results by 3DSTAR1.*

Table 2 *Calculated results by 3DSTAR1 along with analytical solution.*

Time interval (s)	Time to apogee (s)	Apogee height (m)	Time to impact (s)
1.0	7	314.246	16
0.1	7.0	314.3288	16.0
0.01	7.06	314.3479	16.08
0.001	7.063	314.3479	16.087
Analytical solution by equations (30)–(33)	7.0637	314.3485	16.0857

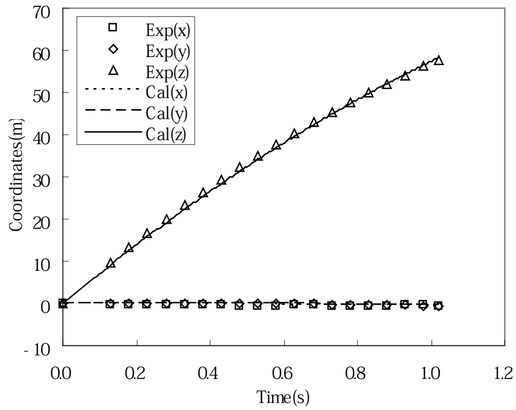

Figure 6 *Observed and calculated trajectories of an airborne No. 3 shell (run 1).*

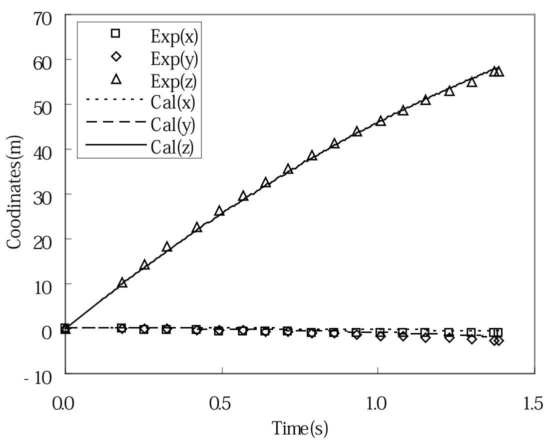

Figure 7 *Observed and calculated trajectories of an airborne No. 3 shell (run 2).*

different directions. These pictures were analyzed and the trajectory of the shell was expressed using three dimensional rectangular coordinates. The results are shown in Figures 6–11.

It is seen from Figures 6–11 that the shells were expelled nearly perpendicularly but moved with a tilt angle at high altitude. This may be the effect of the wind.

The muzzle velocity of the shell was determined from the trajectory of the shell in the air. The results are listed in Table 1. The muzzle velocity tends to increase with increasing mass of lifting charge, but the scatter is large.

Trial and error calculations were carried out using

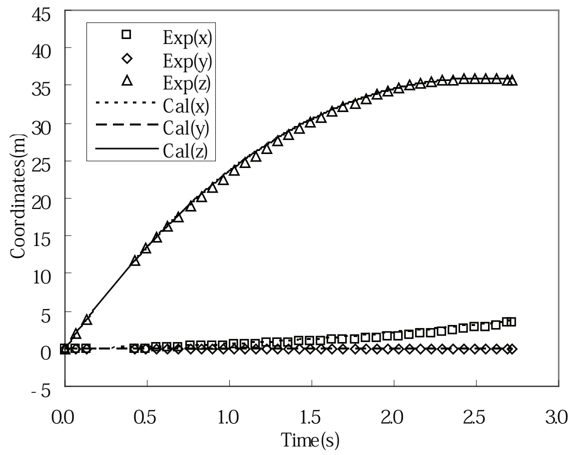

Figure 8 *Observed and calculated trajectories of an airborne No. 3 shell (run 3).*

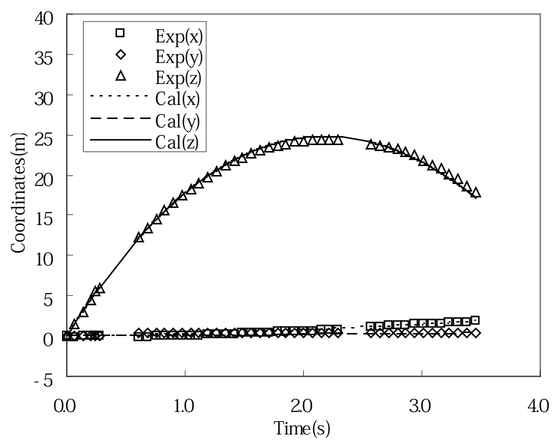

Figure 9 *Observed and calculated trajectories of an airborne No. 3 shell (run 4).*

the 3DSTAR1 code for fitting the drag coefficient, wind speed and wind direction to the observed trajectory of the shell, and the optimal values were obtained. These values are listed in Table 3. The calculated trajectories with these values are also shown in Figures 6–11.

Conclusions

A shot experiment using a No. 3 spherical shell with an illuminant was carried out and three dimensional calculations were performed. The following conclusions were derived:

(1) The three dimensional trajectory of the flying

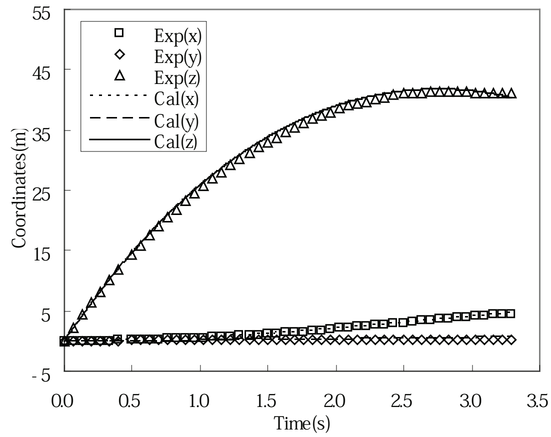

Figure 10 *Observed and calculated trajectories of an airborne No. 3 shell (run 5).*

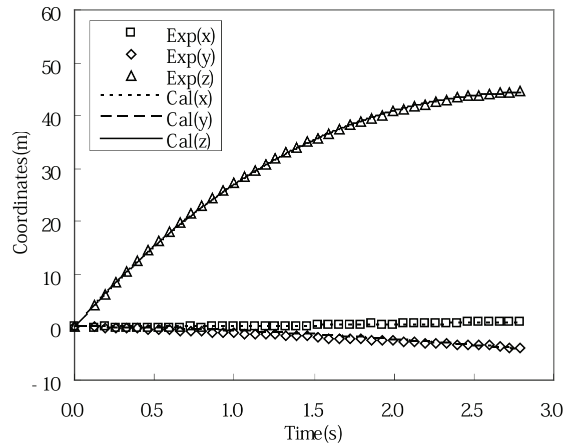

Figure 11 *Observed and calculated trajectories of an airborne No. 3 shell (run 6).*

shell was obtained by tracing the shell with two high-speed video cameras aimed in different directions, and by analyzing the recorded video picture three dimensionally.

(2) The 3DSTAR1 code was developed for calculating the three dimensional trajectory of a flying shell with a high calculation accuracy.

(3) The optimal drag coefficient, wind speed and wind direction were estimated using 3DSTAR1 code by fitting the calculated trajectory to the experimental one for a No. 3 shell.

Acknowledgement

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Table 3 *Muzzle velocity and drag coefficient of the shell, wind speed and wind direction.*

Run	Muzzle velocity ($m\ s^{-1}$)	Drag coefficient	Wind speed ($m\ s^{-1}$)	Angle between directions of wind and coordinate *x* (°)
1	73	0.53	5	225
2	57	0.53	10	−110
3	30	0.99	7	0
4	23	0.71	4	10
5	33	0.67	7	8
6	34	0.63	8	−80

References

1 T. Shimizu, "On Ballistics of Fireworks Shells", *Journal of the Industrial Explosives Society, Japan*, vol. 18, no. 3, 1957, pp. 212–225.

2 T. Shimizu, "On Ballistics of Fireworks Shells (2) Character of the Air Resistance and Effect of Grain Size of the Lifting Charge", *Journal of the Industrial Explosives Society, Japan*, vol. 20, no. 5, 1959, pp. 336–349.

3 K. L. Kosanke and B. J. Kosanke, "Computer Modeling of Aerial Shell Ballistics", *Pyrotechnica*, XIV, 1992, pp. 46–57.

4 M. Iida, S. Hatanaka and N. Suruga, "Shot Experiment of Firework Shell (1)", *Abstracts of Spring Meeting of Japan Explosives Society*, May 26–27, 2005, Otsu, pp. 65–67.

5 D. Eckhardt and H. Andre, "Results and Conclusions from the Investigation of an Accident with a Display Shell", Proceedings of the 5th International Symposium on Fireworks, Naples, Italy, 2000, pp. 85–103.

6 M. Speer, "Reasons for Fuse Failure and Drift Distance of Spherical Fireworks Shells", *Journal of Pyrotechnics*, Issue 17, Summer 2003, pp. 35–52.

7 J. E. Mercer, "Thermodynamics of Black Powder and Aerodynamics of Propelled Aerial Shells", *Journal of Pyrotechnics*, Issue 16, Winter 2002, pp. 37–52.

8 R. L. Schneider and S. C. Schneider, "External Ballistic Calculations for Display Fireworks Launched from Elevated Locations", *Proceedings of the 8th International Symposium on Fireworks*, Shiga, Japan, 2005, pp. 408–415.