Chapter 3 — Problems in Designing Chrysanthemum Shells

In manufacturing chrysanthemum shells, most of the effort of fireworkers has been directed towards obtaining a certain number of ‘petals’ and at the same time obtaining a large flower radius. In this paper the author evaluates the fundamental conditions required to meet these objectives.

The problems are two-fold. The first problem is the initial velocity attained by the stars from the action of the burst charge, that is, the problem deals with the explosion (or bursting) of the shell. The next problem is the motion of the star, which, once initially accelerated, flies in air with a trajectory of some sort and travels a certain distance. In other words, the problem is one of ballistics.

3.1. Factors that Have an Influence on the Initial Velocity of Stars

It is difficult to judge the influence of the various factors without experiments. However, from qualitative considerations they are thought to be as follows. The burst charge begins to burn upon its ignition. When the pressure reaches the breaking pressure, the shell explodes. The stars begin to move as a result of the action of the expanding gas from the explosion. This acceleration ceases when the pressure behind the star becomes equal to the pressure that comes from the air resistance to the motion of star. It is known from interior ballistics[2] that the initial velocity of the star is affected by the following factors: (1) the maximum pressure of breaking the shell, (2) the loading density of the burst charge, (3) the sectional density of the star (the value of the mass of star divided by the star’s largest cross-sectional area), (4) the force of the explosion of the burst charge and (5) the burn rate of burst charge at normal air pressure (i.e., vivacity).

Generally, the combustion reaction of the burst charge is most effective in producing a high pressure when the breaking strength of the shell is large. This imparts a high initial velocity to the stars. It is necessary to increase the strength of the shell for the stars to travel a greater distance.

3.2. Calculation of the Break Strength of a Shell

The breaking resistance of a hollow spherical body (such as a firework shell) is introduced as follows.

Symbols are defined as:

r_i = inner radius $\quad$ P_i = internal pressure

r_a = outer radius $\quad$ P_a = external pressure

r = intermediate radius between r_i and r_a

On the shell, the internal pressure P_i and the external pressure P_a act on each other at the same time. The internal pressure P_i is a force on a unit area of the inside surface acting along the radius and is uniformly distributed on the surface. P_a is the external force that acts on a unit area of surface along the radius of the sphere and is uniformly distributed on the outside surface of the shell (Figure 7).

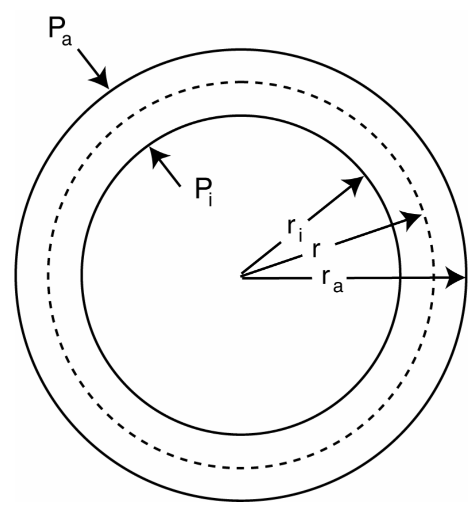

Figure 7. The pressures that act on the paper shell.

The following two assumptions are made about what happens when the sphere of the shell is deformed.

(1) When the sphere is deformed by the action of a force from the outside or inside, the molecules on either surface of the sphere remain on that same surface.

(2) Consequently, after the deformation, the molecules originally on a surface of radius r exist on the deformed surface of radius $r + \Delta r$

From above assumptions, the smallest volume dv in the wall of the paper shell is assumed to be a small spherical hexahedron as indicated by ABCD–EFGH in Figure 8.

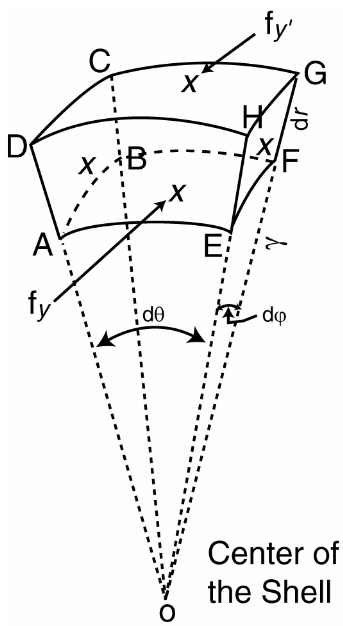

Figure 8. A small spherical hexahedron representing the thickness of the paper shell.

In this case

AB = rdφ = EF

AE = rdθ = BF

Therefore, the volume of the hexahedron is

dv = r^2drdθdφ

When no pressure acts on the surface, the length AO and AD are r and dr. When pressure acts on the surface, these values change to $r + \Delta r$ and dr + d(Δr) [i.e., the both lengths increase: AO increases by Δr, and AD increases by d(Δr)]. Therefore, the deforming ratio of the molecules along the radius is d(Δr)/dφ. When no pressure acts, the length of AE is rdθ, and when the pressure acts it becomes (r +Δr)dθ and it deforms to Δrdθ. Therefore, the deformation in the direction of the tangent is Δrdθ/rdθ = $\Delta r/r$.

Three surfaces at the point A are denoted as:

$$\left.\begin{aligned} \mathrm{ABFE} &= r^2 \mathrm{d}\varphi \mathrm{d}\theta = f_x \\ \mathrm{ADHE} &= r \mathrm{d}r \mathrm{d}\theta = f_y \\ \mathrm{ABCD} &= r \mathrm{d}r \mathrm{d}\varphi = f_z \end{aligned}\right\} \qquad (1)$$

The surfaces of the small volume dv meet at right angles to each other before the deformation. After the deformation the surfaces are at right angles to each other as before. Therefore the volume is not affected by shearing force, but only by internal force at a right angle. The internal forces received by the surfaces f_x, f_y and f_z are denoted as σ_x, σ_y, and σ_z.

The small volume dv changes by the action of σ_x, σ_y, and σ_z so that the unit lengths are deformed, becoming d(Δr)/dr, dr/r, dr/r. Then the internal forces are calculated as:

$$\sigma_x = \frac{E}{m(3-m)} \left\{ \begin{aligned} &(1+m)\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} + (1-m)\frac{\Delta r}{r} + \\ &(1-m)\frac{\Delta r}{r} \end{aligned} \right\}$$

$$\sigma_y = \frac{E}{m(3-m)} \left\{ \begin{aligned} &(1-m)\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} + (1+m)\frac{\Delta r}{r} + \\ &(1-m)\frac{\Delta r}{r} \end{aligned} \right\}$$

That is,

$$\sigma_x = \frac{E}{m(3-m)} \left\{ (1+m)\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} + 2(1-m)\frac{\Delta r}{r} \right\} \qquad (2)$$

$$\sigma_y = \sigma_z = \frac{E}{m(3-m)} \left\{ (1-m)\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} + 2\frac{\Delta r}{r} \right\} \qquad (2')$$

Here E is the elasticity, and m is the value for the volume deformation derived from the deformation coefficient of the length along the direction of the outside pressure. Following the above equations, the interior force has no relation to φ or θ.

The values of the deformation coefficient, $d(\Delta r)dr$, $\Delta r/r$, of the volume dv along the three axes are then determined. For this purpose, first an equilibrium formula of the internal forces on the micro cube is constructed. The interior force on the three surfaces are $\sigma_x f_x$, $\sigma_y f_y$, $\sigma_z f_z$. The area of DCGH, which faces ABFE, is

$$f_{x'} = (r + \mathrm{d}v)^2 \,\mathrm{d}\varphi \mathrm{d}\theta$$

and the force acting on the right angle is

$$\sigma_{x'} = \sigma_x + (\mathrm{d}\Delta r / \mathrm{d}r)\mathrm{d}r$$

Therefore the total interior force acting on:

$$\begin{aligned} f_{x'}\sigma_{x'} &= \left\{ (\sigma_x \left(\frac{\mathrm{d}\sigma x}{\mathrm{d}r} \right) \mathrm{d}r \right\} f_{x'} \\ &= \left\{ (\sigma_x \left(\frac{\mathrm{d}\sigma x}{\mathrm{d}r} \right) \mathrm{d}r \right\} (r + \mathrm{d}r)^2 \,\mathrm{d}\varphi\theta \end{aligned} \tag{3}$$

The surface BCGF is equal to the area of ADHE,

$$f_y = f_{y'} = r\mathrm{d}r\mathrm{d}\theta$$

and the interior right angle force σ_y is equal to the right angle force σ_y on the surface of ADHE. The value of σ_y from equation 2 is independent of θ and a function only of r and the surface of radius r, the values of σ_y should be equal in every point. Therefore, it becomes

$$\sigma_y f_y = \sigma_{y'} f_{y'}$$

In the same is true for the surface EFGH:

$$\sigma_z f_z = \sigma_{z'} f_{z'}$$

Therefore on the surface of the micro cube there arise six forces at interior right angle with no shearing force. These forces should balance each other as seen in Figure 9.

Figure 9 shows the surface ABCD, but the other surfaces are in the same state. For the equilibrium of these forces the following rules apply:

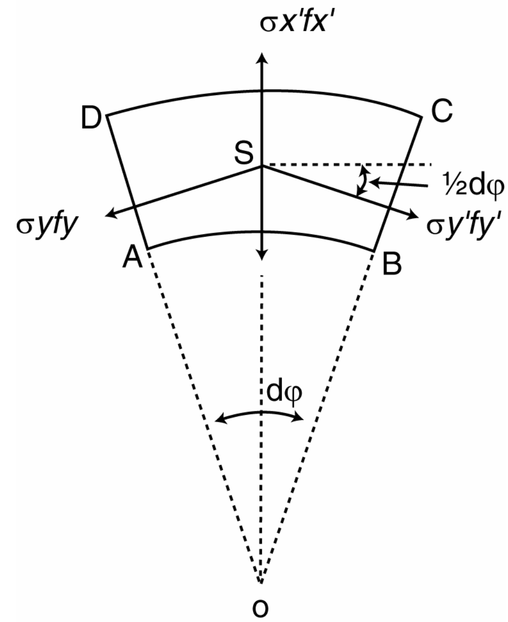

Figure 9. Equilibrium of internal forces in the thickness of the paper shell.

(1) When the forces are projected onto the three right angle axes, the sum of the components of force is zero.

(2) The sum of moments along the three right angle axes is zero.

The equilibrium equations of internal forces are as follows:

for the direction of X axis:

$$\begin{aligned} \Sigma X = \sigma_{x'} f_{x'} &- \sigma_z f_z - \sigma_y f_y \sin(\tfrac{1}{2}\mathrm{d}\varphi) \\ &- \sigma_{y'} f_{y'} \sin(\tfrac{1}{2}\mathrm{d}\varphi) - \sigma_z f_z \sin(\tfrac{1}{2}\mathrm{d}\theta) \\ &\sigma_{z'} f_{z'} \sin(\tfrac{1}{2}\mathrm{d}\theta) \\ &= 0 \end{aligned} \tag{4}$$

for the direction of Y axis:

$$\begin{aligned} \Sigma Y &= \sigma_{y'} f_{y'} \cos(\tfrac{1}{2}\mathrm{d}\varphi) - \sigma_y f_y \cos(\tfrac{1}{2}\mathrm{d}\varphi) \\ &= 0 \end{aligned} \tag{5}$$

the direction of the Z-axis should be the same as for the Y-axis.

From equation 5 we have $\sigma_{y'} f_{y'} = \sigma_y f_y$. This is the relationship described earlier. In equation 4 relationships (1) and (3) are substituted, and we have

$$2(\sigma_x - \sigma_y) + \sigma_x \frac{\mathrm{d}r}{r} +$$

$$\frac{\mathrm{d}\sigma_z}{\mathrm{d}r}\left(r + 2\mathrm{d}r + \frac{\mathrm{d}r}{r}\right) = 0$$

In this equation the multiples of dr can be omitted to give:

$$2(\sigma_x - \sigma_y) + r\frac{\mathrm{d}\sigma_x}{\mathrm{d}r} = 0 \tag{6}$$

Equation 2 is differentiated with respect to r and multiplied on both sides by r:

$$r\frac{\mathrm{d}\sigma_x}{\mathrm{d}r} = \frac{E}{m(3-m)}\left\{\begin{array}{l}(1+m)\cdot r\cdot\frac{\mathrm{d}^2\Delta r}{\mathrm{d}r^2} + \\ 2(1-m)\frac{\mathrm{d}\Delta r}{\mathrm{d}r} - \\ 2(1-m)\frac{\Delta r}{r}\end{array}\right\} \tag{7}$$

When equations 2, 2' and 7 are incorporated into equation 6, we have

$$\frac{\mathrm{d}^2(\Delta r)}{\mathrm{d}r^2} + 2\cdot\frac{1}{r}\cdot\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} - 2\frac{\Delta r}{r^2} = 0 \tag{8}$$

Now we put a special solution for equation 8 as

$$\Delta r = r^n \tag{9}$$

and the value of n is obtained:

$$\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} = nr^{n-1} \quad , \quad \frac{\mathrm{d}^2(\Delta r)}{\mathrm{d}r^2} = n(n-1)r^{n-2} \tag{10}$$

These are introduced into (8) and we have

$$n(n-1)r^{n-2} + 2nr^{n-2} - 2r^{n-2} = 0$$

The value of r is not zero, and it must be

$$n(n-1) + 2n - 2 = 0$$

Therefore

$$(n-1)(n+2) = 0$$

Solving this, we have

$$n_1 = 1 \tag{11}$$

or

$$n_2 = -2 \tag{12}$$

When these are introduced into equation 9, we have two linearly independent special solutions:

$$\Delta r_1 = r^{n_1} \quad , \quad \Delta r_2 = r^{n_2}$$

Therefore, two constants A and B are introduced and we have a general solution of equation 8:

$$\Delta r = Ar^{n_1} + Br^{n_2} \tag{13}$$

Next the formula to obtain the equivalent stress is established. First, the constants A and B are obtained as follows:

From equation 2

$$\sigma_x = e\left\{(1+m)\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} + 2(1-m)\frac{\Delta r}{r}\right\} \tag{14}$$

where

$$e = \frac{E}{m(3-m)} \tag{15}$$

Equation 13 is differentiated with respect to r

$$\frac{\mathrm{d}(\Delta r)}{\mathrm{d}r} = n_1 Ar^{n_1-1} + Br^{n_2-1} \tag{16}$$

Equation 13 is divided by r

$$\frac{\Delta r}{r} = Ar^{n_1-1} + Br^{n_2-1} \tag{17}$$

Equations 16 and 17 are introduced into equation 14

$$\begin{aligned}\sigma_x &= e\left\{(1+m)n_1 r^{n_1-1}A + (1+m)n_2 r^{n_2-1}B\right\} \\ &\quad + 2(1-m)r^{n_1-1}A + 2(1-m)r^{n_2-1}B \\ &= d\left(A\alpha r^{n_1-1} + B\beta r^{n_2-1}\right)\end{aligned} \tag{18}$$

In equation 18

$$\left.\begin{aligned}\alpha &= (1+m)n_1 + 2(1-m) \\ \beta &= (1+m)n_2 + 2(1-m)\end{aligned}\right\} \tag{19}$$

The value of the normal internal force is equal to the internal force on the internal surface and to the outside pressure P_a on the outside of shell with. Both of the pressures on the inside and outside of the shell press the thickness of the shell along the direction of the radius, and when $r = r_i$, $\sigma_x = -P_i$ and when $r = r_a$, and $\sigma_x = -P_a$. Therefore from equation 18

$$-P_i = e\left(A\alpha r_i^{n_1-1} + B\beta r_i^{n_2-1}\right) \tag{20}$$

$$-P_a = e\left(A\alpha r_a^{\,n_1-1} + B\beta r_a^{\,n_2-1}\right) \tag{21}$$

Equation 20 is multiplied by $r_i r_a^{n_1}$, equation 21 is multiplied by $r_a r_i^{n_2}$, and then they are subtracted from each other:

$$P_a r_a r_i^{n_1} - P_i r_i r_a^{n_1} = e\beta B\left(r_i^{n_2} r_a^{n_1} - r_a^{n_2} r_i^{n_1}\right)$$

From the above:

$$B = \frac{1}{e\beta}\frac{P_a r_a r_i^{n_1} - P_i r_i r_a^{n_1}}{r_i^{n_2} r_a^{n_1} - r_a^{n_2} r_i^{n_1}} \tag{22}$$

With the same calculation, multiplying equation 20 by $r_i r_a$, and equation 21 by $r_a r_i^{n_2}$ and subtracting from each other we have:

$$P_a r_a r_i^{n_1} - P_i r_i r_a^{n_1} = e\alpha A\left(r_i^{n_1} r_a^{n_2} - r_a^{n_1} r_i^{n_2}\right)$$

From the above:

$$A = \frac{1}{e\alpha}\frac{P_a r_a r_i^{n_2} - P_i r_i r_a^{n_2}}{r_i^{n_1} r_a^{n_2} - r_a^{n_1} r_i^{n_2}} \tag{23}$$

When equations 22 and 23 are introduced into equation 17 we have:

$$\frac{\Delta r}{r} = \frac{1}{e}\cdot\frac{1}{\left(r_i^{n_2} r_a^{n_1} - r_a^{n_2} r_i^{n_1}\right)} \left\{\begin{array}{l} \left(P_a r_a r_i^{n_1} - P_i r_i r_a^{n_1}\right)\dfrac{r^{n_2-1}}{\beta} \\ -\left(P_a r_a r_i^{n_2} - P_i r_i r_a^{n_2}\right)\dfrac{r^{n_1-1}}{\alpha} \end{array}\right\} \tag{24}$$

Next, when the shell is broken, it is sufficient to consider the deformation resistance along the direction of the tangent. This force is denoted as S_y.

$$S_y = E\cdot\frac{\Delta r}{r}$$
$$= \frac{m(3-m)}{r_i^{n_2} r_a^{n_1} - r_a^{n_2} r_i^{n_1}} \left\{\begin{array}{l} \left(P_a r_a r_i^{n_1} - P_i r_i r_a^{n_1}\right)\dfrac{r^{n_2}}{\beta} \\ -\left(P_a r_a r_i^{n_2} - P_i r_i r_a^{n_2}\right)\dfrac{r_i^{n_1-1}}{\alpha} \end{array}\right\} \tag{25}$$

The force comes only from the inner pressure. Therefore $P_a = 0$.

$$S_{y_i} = \frac{m(3-m)P_i r_i}{r_i^{n_2} r_a^{n_1} - r_a^{n_2} r_i^{n_1}}\left(\frac{r_a^{n_2}}{\alpha}\cdot r^{n_1-1}\,\frac{r_a^{n_1}}{\beta}\,r^{n_2-1}\right) \tag{26}$$

An example of the calculation is as follows:

$m = 1/3$

(a) r_i = 68 mm, r_a = 76 mm (corresponds to chrysanthemum No. 5), and

(b) r_i = 53 mm, r_a = 61 mm (corresponds to chrysanthemum No. 4). The values of S_{y_i} / P_i are calculated, and the results are shown in Table 1 and Figure 10.

Table 1. Calculated Values of S_{y_i} / P_i.

(a)		(b)	
r (mm)	S_{y_i} / P_i	r (mm)	S_{y_i} / P_i
68	3.15	53	2.57
70	2.96	55	2.37
72	2.79	57	2.19
74	2.63	59	2.04
76	2.49	61	1.90

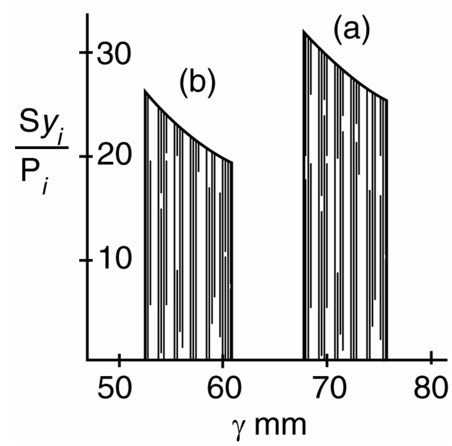

Figure 10. Calculated values of S_{y_i} / P_i. (See Table 1).

From the results of the calculations in Table 1 and Figure 10, we see that the resistance to the deformation is the greatest at the interior surface. Therefore, when the value S_{y_i} is greater than the internal resistance force, the shell is

Table 2. The Critical Internal Breaking Pressure P_0 (kg/cm^2) for Internal Diameter γ_i and Thickness of Shell δ.

γ_i (mm)	δ 2.0	4.0	6.0	8.0	10.0
25	46.4	85.7	119.6	146.3	169.3
50	24.2	46.5	66.9	85.6	102.9
75	16.4	31.9	46.5	60.4	73.0
100	12.6	24.3	35.6	46.3	57.0
125	10.0	19.2	28.6	38.0	46.5
150	7.9	16.1	23.9	32.0	38.9

broken. Therefore, we can obtain the critical internal breaking pressure when we set $r = r_i$, $S_{y_i} = J_i$. The critical limit of the internal breaking pressure is given by equation 27.

$$P_0 = \frac{J_z}{m(3-m)} \frac{r_a^{n_1} r_i^{n_2} - r_i^{n_1} r_a^{n_2}}{\frac{r_a^{n_2} r_i^{n_1}}{\alpha} - \frac{r_a^{n_1} r_i^{n_2}}{\beta}} \tag{27}$$

Example: $J_z = 210$ kg/cm^2 (internal force of resistance of Japanese paper), the internal radius of the shell $r_i = 25 - 150$ mm, the thickness of the shell = 2 – 10 mm: the calculation results are shown in Table 2 and Figure 11. Then, for various internal radii the thickness and the difference of the thickness for a definite breaking force are shown in Table 3.

According to Table 3 the values of $\Delta\delta$ are constant for a definite interior pressure, even when the values of r_i change. Consequently, when the internal breaking pressure is increased, the strength of the shell should be increased in proportion to the radius of the shell. This coincides with the experience of fireworkers described in Chapter 2.

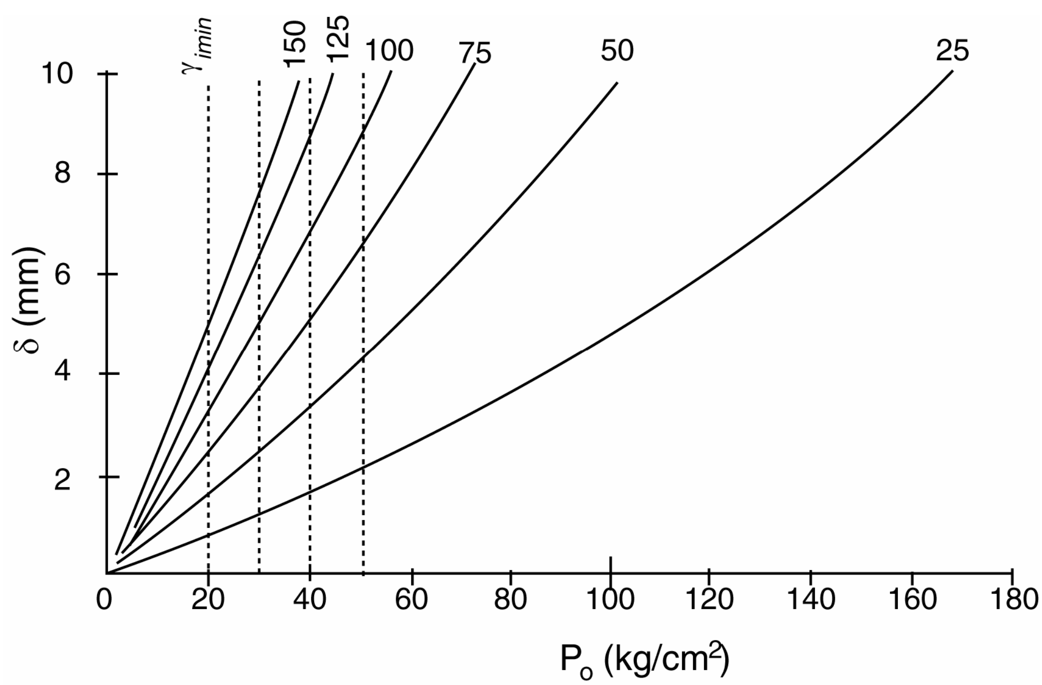

Figure 11. Graphs for Table 2.

Table 3. Internal Diameter γ_i and Thickness of Shell δ and $\Delta\delta$ for Critical Pressure γ_i.

γ_i (mm)	P_0 (kg/cm^2)							
	20		30		40		50	
	δ	$\Delta\delta$	δ	$\Delta\delta$	δ	$\Delta\delta$	δ	$\Delta\delta$
25	0.8		1.3		1.7		2.2	
		0.9		1.2		1.7		2.1
50	1.7		2.5		3.4		4.3	
		0.8		1.3		1.7		2.1
75	2.5		3.8		5.1		6.4	
		0.8		1.2		1.6		2.2
100	3.3		5.0		6.7		8.6	
		0.8		1.2		1.7		—
125	4.1		6.2		7.8		—	
		0.9		1.3				—
150	5.0		7.5		—		—	

It should be noted that equation 27 applies to a theoretically uniform material that forms a hollow sphere. However, shells are not uniform, because they are made of paper. The calculations just described must be expected to have some unknown problems when they are applied to the design of a shell. The value for "*m*" (Poisson's number) should be determined experimentally.[2] None the less, this method should be useful as a rough guide to designing a chrysanthemum shell.

3.3. The Effect of the Type of Paper on the Strength of the Shell

Experience teaches us nothing except the principle of *per sun*, which means the strength of the shell is proportional to the folds of paper pasted on the shell without consideration to the diameter of the shell. That is to say, for example, eight folds *per sun* of Kraft paper on the 5 sun (6 inch) shell gives the same breaking strength as eight folds per sun on the 10-sun (12-inch) shell. However, it does not teach us about the quality of the paper.

In Japan we use Japanese paper (Kozo Paper) or Kraft paper. Japanese paper is soft and before use, two or three sheets are pasted together into one sheet. It is easy to work with, but its high price prevents using it. Kraft paper is less convenient to use than the Japanese paper, but it is less expensive, so at present many fireworkers use Kraft. In this paper both types of paper are studied.

3.4. Mass per Unit Cross Sectional Area of a Spherical Star (Sectional Density)

The equation for the start of a star's motion is as follows:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\sigma\left(P_b-P_f\right) \qquad (28)$$

where m is the mass of a star, σ is the area of the greatest cross section of the star, $\mathrm{d}^2x/\mathrm{d}t^2$ is the acceleration, P_f is the resistant force per unit area (pressure) that acts on the front of the star, and P_b is the pushing pressure that acts on the rear of the star.

Integrating equation 28 we have:

$$\Delta v=\frac{\sigma}{m}\int_0^t\left(P_b-P_t\right)\,\mathrm{d}t,\ \ \Delta v=v-v_t=0 \qquad (29)$$

This shows that the change in the velocity of the star is inversely proportional to the value of the sectional density m/σ, which depends on the true density and the size of the star. When the true density of the star is constant, the value of m/σ increases or decreases in proportion to the diameter of the star. The true density is generally 1.3–1.6 grams per cubic centimeter, and it is difficult to obtain larger or smaller values. The smaller the sectional density, the larger the initial velocity of the star. However, after the driving force of the burst charge diminishes, the star is greatly decelerated by the air resistance. This must be remembered when designing the chrysanthemum.

3.5. The Compositions of the Burst Charge

The compositions that have long been used in Japan were mixtures of potassium chlorate and hemp charcoal, which were called the *haisan* explosives. (The meaning of the word *haisan* or H3 comes from the mixing ratio of hemp charcoal 3 parts to potassium chlorate 10 parts). For safety reasons some *haisan* are mixed with potassium nitrate. It must be remembered not to mix sulfur or red phosphorus with the *haisan*. In ordinary firework factories, they use Black Powder that contains sulfur. In spite of the danger of mixing sulfur with chlorate, the reasons for using *haisan* are:

1) it is very quick burning,
2) it provides a large explosive force,
3) it requires only a simple manufacturing process, when compared to black powder, and
4) it is the cheapest of all.

In this study three compositions are studied:

1) Black Powder,
2) potassium perchlorate composition, and
3) potassium chlorate composition (*haisan*).

3.6. The Force of Explosives *f*

Following the theory of internal ballistics, the energy output E of a propellant is denoted as

$$E=\frac{fw}{\gamma-1} \qquad (30)^{[2]}$$

where w is the mass of the explosive charge, f is the energy output per unit mass of explosive and γ is the average adiabatic expansion coefficient of the gas, typically 1.25.

The value of f in joules/kilogram is given by nRT where n is the number of moles of gas produced by 1 kilogram of explosive, R is the gas constant and T is the explosion temperature in kelvin. If V_0 is the volume of gas produced by the explosion of 1 kg of explosive, reduced to 273.15 K and 1 atmosphere pressure, and if the energy is expressed as kilograms weight x decimeters, the value of f is given by 0.3782 x V_0 x T, units decimeters. This is how f will be calculated in Chapter 4.

When burnt gas of the burst charge projects a star of mass m with the initial velocity V we have:

$$\tfrac{1}{2}mV^2=\frac{fw}{\gamma-1}\cdot\sigma R$$

Here, R denotes the effective fraction of the charge that acted on the unit cross sectional area σ of the star, and it is called the efficiency. From the above equation:

$$\bar{V} = \sqrt{\frac{2}{\gamma - 1}} \cdot (m)^{1/2} (fw)^{1/2} R^{1/2} \sigma^{1/2}$$
$$= K \cdot E^{1/2} \cdot R^{1/2} \cdot \left(\frac{m}{\sigma}\right)^{-1/2} \qquad (31)$$

The value of *fw* is independent of *R*, and the initial velocity is proportional to the square root of *fw*. This should be confirmed by experiments.

3.7. The Vivacity *A* of Burst Charge

The vivacity of the burst charge has an influence on the initial velocity of a star with the effect of the breaking strength of the shell. Usable quantitative data concerning this problem have not yet been obtained. The core substance of the burst charge should be also selected to give good values for the vivacity. This is the important goal of this chapter.

3.8. Loading Density

The construction of the burst charge of the chrysanthemum is almost uniform among fireworkers, and the values are 0.28–0.32 kg/dm^3. The values are obtained from the mass of the burst charge divided by the space occupied by the stars and the volume of supporting material (core). The relation of the breaking strength and the loading density is denoted by the equation of Noble and Abel:[1]

$$P_0 = f \frac{\Delta Z_0}{1-\eta} = \frac{fZ_0}{\frac{1}{\Delta} - \eta} \qquad (32)$$

where Δ is loading density, Z_0 is the burning ratio of the burst charge at the breaking of the shell, namely the ratio of burnt burst charge to initial mass of the burst charge, η the covolume. The breaking pressure P_0 is thought to be constant, and Z_0 decreases with the increase of Δ.

3.9. Factors Concerning the Velocity of the Stars

Air decelerates moving stars. According to fluid mechanics, air resistance acting on a star is given by:

$$D = k' \rho v^2 \sigma \left(\frac{\mu}{\rho \ell \bar{v}}\right)^{\delta} = k' \rho v^2 \sigma \left(\frac{1}{Re}\right)^{\delta} \qquad (33)$$

where ρ is the density of the air, μ is the coefficient of viscosity of air, v is the velocity of the star, σ is the cross sectional area of the star, ℓ is the diameter of the star, and *Re* is the Reynolds number:

$$Re = \frac{\rho \ell v}{\mu} = \rho v^2 \div \frac{\mu v}{\ell} \propto \frac{\text{Force of inertia}}{\text{Force of viscosity}} \qquad (34)$$

The fluid is air. Therefore, the force of inertia is much greater than the force of viscosity. The value of the Reynolds number is therefore also very large. From equation 33

$$\left(\frac{1}{Re}\right)^{\delta} = X$$

From this the value of δ is obtained:

$$\delta = -\frac{\log X}{\log Re} \qquad (35)$$

As the value of *Re* approaches infinity, the value of δ approaches zero.

The terms

$$k' \left(\frac{\mu}{\rho \ell \bar{v}}\right)^{\delta} = k' \left(\frac{1}{Re}\right)^{\delta}$$

can be replaced by another constant, *k*.

The motion equation for a star flying in a horizontal direction is therefore

$$m \frac{d^2 x}{dt^2} = -D = -k \rho v^2 \sigma \qquad (36)$$

For a star that flies in a vertical direction the motion equation is:

$$m \frac{d^2 y}{dt^2} = -k u^2 \sigma - mg \qquad (37)$$

where g is the acceleration due to gravity. The above two equations are summarized as follows:

$$\left.\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{k}{p'(t)}v^2 \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= -\frac{k}{p'(t)}u^2 - g\end{aligned}\right\} \qquad (38)$$

where p' is the sectional density of the star, i.e the mass of the star divided by the cross sectional area of the sphere at that time.

3.10. The Relation of Physical Conditions and the Visual Beauty

As described earlier, the uniformity and the straightness of the motion of star are most important for the beauty of the chrysanthemum. The trajectory of stars should be as straight as possible and the trace of the moving stars should be uniform with respect to time and space.

The chrysanthemum must develop the beauty of the star light as a quantitative effect. Therefore the number of stars contained in a shell should be as large as possible. If the diameter of the shell is limited, the stars should be as small as possible. However, in this case the density of the stars becomes smaller as the diameter of the chrysanthemum becomes smaller. These factors should be examined later.

3.11. Calculating the Number of Stars in a Round Shell

The number of round stars that should be contained in a spherical shell is calculated as follows. The diameter of the shell is denoted as *D*, and the diameter of the stars as *d*. Now six stars are arranged in a flat layer as shown in Figure 12. Actually, there are seven stars, the centers of which are o, a, b, c, d, e, and f.

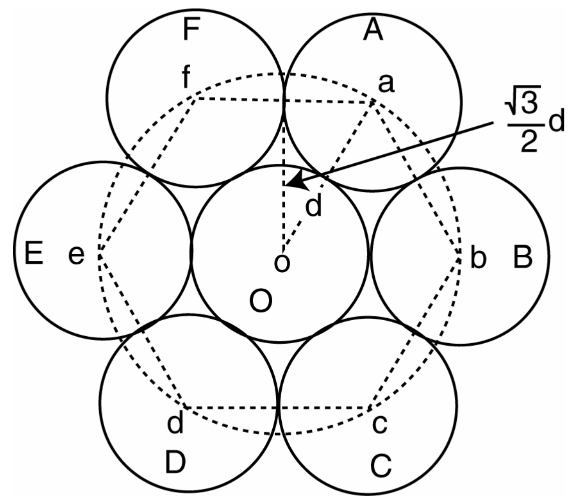

Figure 12. Basic calculation of the number of stars arranged in a shell.

The area of the hexagon abcdef is:

$$S = \frac{\sqrt{3}}{4}d^2 \times 6$$

This area contains the center star "o" and 1/3 of a hexagon ABCDEF. Therefore this area is the space that can contain 3 stars (1 + (1/3) × 6 = 3) and the area per star is:

$$\varsigma = \tfrac{1}{3}S = \frac{\sqrt{3}}{2}d^2$$

However, the centers of the stars are not on a flat surface, but on a sphere, so the above formulas only show the rough calculation.

The distance from the center of shell to the contact point of each star is ½(*D*–*d*) cos θ, where θ is the angle of a star from the center of the shell. The area of the sphere that contains the contact points of the stars is:

$$\begin{aligned}\varsigma' &= \pi(D-d)^2\cos^2\theta \\ &= \pi(D-d)^2(1-\sin^2\theta) \\ &= \pi(D-d)^2\left\{1-\frac{d^2}{(D-d)^2}\right\}\end{aligned}$$

Therefore, the number of stars *n* is denoted as

Table 4. The Number of Main Stars along the Inside of a Shell.

Shell Size (inch)	Internal Diam. of Shell (mm)	Diameter of Star (mm)	Number of Stars (calculated)	Number of Stars (empirical)
3-1/2	78	9.9	201	200
5	104	12.7	220	220
6	132	15.4	245	260
7	162	17.6	291	280
8	190	19.3	336	320
9-1/2	216	20.7	411	330
12	266	23.1	437	420

$$n = k\frac{\varsigma'}{\varsigma} = k\frac{\pi(D-d)^2\left\{1-d^2/(D-d)^2\right\}}{\sqrt{3}d^2/2}$$
$$= k\frac{2}{\sqrt{3}}\pi\frac{D}{d}\left(\frac{D}{d}-2\right)$$

where k is an adjustment factor based on experience. If k is set to 1.20, we have:

$$n = 4.35\frac{D}{d}\left(\frac{D}{d}-2\right) \qquad (39)$$

This is a formula to calculate the number of spherical stars to be contained in a shell. A comparison between the calculated and empirical number of stars are shown in Table 4.